MATHEMATICS

1_. $f'(c) = \frac{f(2)-f(0)}{2-0}$ ⇒ $f'(c) = 1$ ⇒ $\frac{\pi}{2}\sin(\pi c) = 1, -2(1-c) = 1$

⇒ $\sin \pi c = \frac{2}{\pi}, c = \frac{3}{2}$ ⇒ three values of c

2_. $D(x) = x^2 + f^2(x)$ ⇒ $D'(x) = 2x + 2f(x) f'(x)$
D(x) is minimum ⇒ $D'(a) = 0$ ⇒ $2a + 2f(a) f'(a) = 0$

3_. $f(a) \geq -f(b)$ ⇒ $f(a) \geq f(-b)$ ⇒ $a \geqq -b$ ⇒ $a + b \geq 0$

4_. $-2 \leq r^4 - 2r^2 + 3 \leq 2$ ⇒ $(r^2-1)^2 + 2 \leq 2$ ⇒ $r^2 - 1 = 0$ ⇒ $r = \pm 1$
and $\sin\theta = 1$ ⇒ $\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$

5_. Let $h(x) = f(x) + x$ ⇒ h(x) is irrational for all $x \in R$
But h(x) is continuous
So h(x) can take single irrational value say k
⇒ $h(x) = k$ ⇒ $f(x) = k - x$ ⇒ $f(-k) = 2k$
But this is contradiction to given condition in question

6_. $\frac{d}{dx}(\log_a x) = \frac{d}{dx}(a^x) = 1$ ⇒ $a^x \log a = \frac{1}{x \log a} = 1$

⇒ $x = \log_a e$ and $a^x = \frac{1}{\log a}$ ⇒ $e = \frac{1}{\log a}$ ⇒ $a = e^{1/e}$

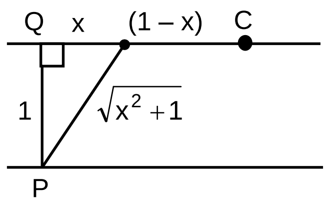
7_. Infection points are $x = \frac{\pi}{2}, \frac{3\pi}{2}$; Tangents are $y = -x + \frac{\pi}{2}$, $y = x - \frac{3\pi}{2}$

Area of triangle = $\frac{1}{2} \times \frac{\pi}{2} \times \pi = \frac{\pi^2}{4}$

8. $f(f(x)) = x + 1$ ⇒ $f(f(f(x))) = f(x) + 1$
⇒ $f(x + 1) = f(x) + 1$ ⇒ $f(x + 1) > f(x)$ ⇒ f can not be decreasing

9. $f''(x) - 6f'(x) > 6$ ⇒ $e^{-6x}(f''(x) - 6f'(x)) > 6e^{-6x}$
⇒ $\frac{d}{dx}(e^{-6x} f'(x) + e^{-6x}) > 0$ ⇒ $e^{-6x}(f'(x) + 1)$ is increasing
also $h(x) = e^{-6x}[f'(x) + 1] = 0$ at $x = 0$ ⇒ $h(x) \geq 0$ ⇒ $f'(x) + 1 \geq 0$
⇒ $[f(x) + x]' \geq 0$ ⇒ $g'(x) \geq 0$ so g(x) is increasing

10. T = D/S ⇒ $T = \frac{\sqrt{x^2+1}}{2} + \frac{1-x}{3} = f(x)$ (say)



$f'(x) = 0$ ⇒ $x = \frac{2}{\sqrt{5}}$ for minima ⇒ f(x) = minimum time = 0.7 hr approximately

11. Let number of additional machines installed = x

$P(x) = (50 + x)(1000 - 10x) \Rightarrow P'(x) = 0 \Rightarrow x = 25$

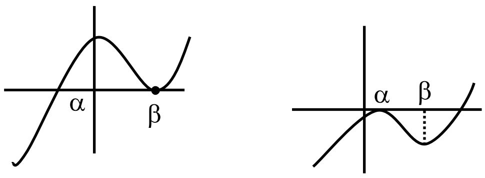
12. Let $\frac{ax^2}{c^3} = \sin\theta$ and $\frac{by^2}{c^3} = \cos\theta \Rightarrow xy = \sqrt{\frac{c^6 \sin\theta\cos\theta}{ab}} \Rightarrow (xy)_{max.} = \sqrt{\frac{c^6}{2ab}}$

13. $S = \sqrt{x^2 + y^2} = \sqrt{x^2 + \frac{1}{x}} \Rightarrow \frac{ds}{dx} = 0 \Rightarrow x = \left(\frac{1}{2}\right)^{1/3}$

14. Let 'x' be the fraction $\Rightarrow$ $y = x - x^n$ has to be maximum

$\Rightarrow \frac{dy}{dx} = 0 \Rightarrow x = \left(\frac{1}{n}\right)^{\frac{1}{n-1}}$ & $\frac{d^2y}{dx^2} < 0$

15.



16. $f(x) = \sin x - \cos x + \sin^{-1}x - \cos^{-1}x$

$\Rightarrow f'(x) = \cos x + \sin x + \frac{2}{\sqrt{1-x^2}} \Rightarrow f'(x) = \sqrt{2}\,\sin\left(x + \frac{\pi}{4}\right) + \frac{2}{\sqrt{1-x^2}}$

As $\frac{2}{\sqrt{1-x^2}} > 2$ so $f'(x) > 0$ for $x \in [-1, 1]$

Also $f(-1) < 0$ and $f(1) > 0$ so one solution

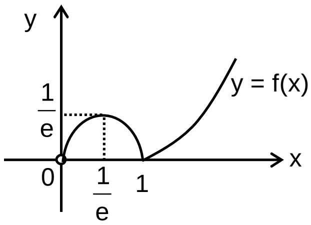
17. Let $g(x) = f(x)\cos x \Rightarrow g'(x) \le 0 \Rightarrow g\left(\frac{5\pi}{3}\right) \le g\left(\frac{\pi}{2}\right)$

$\Rightarrow g\left(\frac{5\pi}{3}\right) \le 0 \Rightarrow \frac{1}{2} f\left(\frac{5\pi}{3}\right) \le 0 \Rightarrow f\left(\frac{5\pi}{3}\right) = 0$

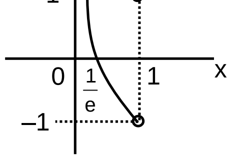
18. $x^2 + 6x + 11 > 0 \quad \forall x \in R$

$f\left(\frac{\pi}{6}\right) = 0,\ f\left(\frac{\pi^+}{6}\right) = (0^+)^n > 0,\ f\left(\frac{\pi^-}{6}\right) = (0^-)^n > 0$

19_*. $f(x) = \begin{cases} -x\ell nx, & 0 < x \le 1 \\ x\ell nx, & x \ge 1 \end{cases}$

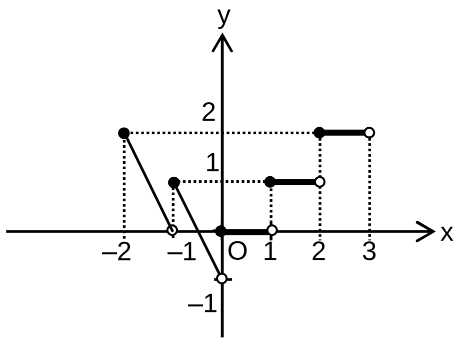


$\Rightarrow f'(x) = \begin{cases} -1 - \ell nx, & 0 < x < 1 \\ 1 + \ell nx, & x > 1 \end{cases} \Rightarrow f''(x) = \begin{cases} -\frac{1}{x}, & 0 < x < 1 \\ \frac{1}{x}, & x > 1 \end{cases}$



By LMVT, $\frac{f'(x+2) - f'(x)}{2} = f''(c)$. As $f''(c) < 1 \Rightarrow f'(x + 2) - f'(x) < 2$

20_*. $f(x) = \begin{cases} -2x-2, & -2 < x < -1 \\ -2x-1, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}$



21_*. $f'(x) \ge 0 \Rightarrow 2e^x - (a^2 - 5a + 6)\, e^{-x} + 10a - 2a^2 - 11 \ge 0$

$\Rightarrow e^{-x}(2e^{2x} - (a^2 - 5a + 6) + (10a - 2a^2 - 11)e^x) \ge 0$

$\Rightarrow e^{2x} + (5a - a^2 - \frac{11}{2})e^x + (5a - a^2 - 6)\left(\frac{1}{2}\right) \ge 0$

$\Rightarrow (e^x + 5a - a^2 - 6)\left(e^x + \frac{1}{2}\right) \ge 0$

$\Rightarrow a^2 - 5a + 6 \le e^x \Rightarrow a^2 - 5a + 6 \le 0 \Rightarrow 2 \le a \le 3$

22_*. f(0) = 2 is maximum value

23_*. Take counter examples (A) $f(x) = x^3 - x$ (C) $f(x) = e^x$ (D) $f(x) = x^3$ at x = 0

24_*. $f_n(x)$ must be quadratic $\Rightarrow n \ne 1$

$f_n(x)$ has maxima $\Rightarrow 2 + (-2)^n < 0 \Rightarrow$ n can be odd

25_*. $\sin x_0 = cx_0$ and $\frac{d}{dx}(\sin x)_{x_0} = \frac{d}{dx}(cx)_{x_0} \Rightarrow \cos x_0 = c \Rightarrow \tan x_0 = x_0$

26_*. Let $g(x) = \log(1 + f^2(x)) - x$; $(a, b) \to R \Rightarrow g'(x) = \frac{2f(x)f'(x) - 1 - f^2(x)}{1 + f^2(x)} \ge 0$

$\Rightarrow \lim_{x \to b^-} g(x) \ge \lim_{x \to a^+} g(x) \Rightarrow 1 - b \ge -a \Rightarrow b - a \le 1$

27^*. $f(x) = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^3 - 3.\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) - 4.\left[\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 - 2\right]$

$\sqrt{x} + \frac{1}{\sqrt{x}} = t \Rightarrow f(x) = t^3 - 4t^2 - 3t + 8,\ t \in [2, \infty)$

Let f(x) = g(t) $\Rightarrow g'(t) = (t - 3)(3t + 1) \Rightarrow g(t) \in [g(3)\ g(\infty)) \Rightarrow f(x) \in [-10, \infty)$

Also $f'(x) = (3t^2 - 8t - 3)\left(\frac{x - 1}{2x\sqrt{x}}\right) \Rightarrow f'(1) = 0$

28^*. Since a, b, c, d are in G.P. so $a \ne 0$

Further $f'(x) = 3ax^2 + 2bx + c$

$D = 4b^2 - 12ac = 4b^2 - 12b^2 = -8b^2 < 0$

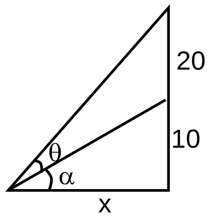
$\Rightarrow f'(x) > 0$ if $a > 0$ and $f'(x) < 0$ if $a < 0 \Rightarrow f(x)$ is monotonic $\Rightarrow f(x) = 0$ has only one real root.

Now, $f''(x) = 0 \Rightarrow 6ax + 2b = 0 \Rightarrow x = \frac{-b}{3a}$ so one root.

29^*. $\tan(\theta + \alpha) = 30/x$ ...(i)

$\tan \alpha = 10/x$ ...(ii)

$$\frac{\tan\theta + \tan\alpha}{1 - \tan\theta \ \tan\alpha} = \frac{30}{x} \quad \Rightarrow \quad \tan\theta = \frac{20x}{x^2 + 300} = f(x)$$



Now $\theta \to \max \quad \Rightarrow \quad \tan\theta \to \max \quad \{\because \theta < 90°\}$
$f'(x) = 0 \quad \Rightarrow \quad x = 10\sqrt{3} \quad \Rightarrow \quad \theta = 30°$

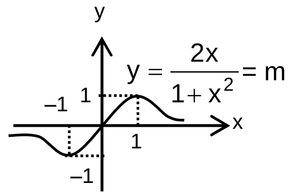
30^*. Let $f(x) = 10 + \int_0^x e^{x^2} dx \Rightarrow f(1) = 10 + \int_0^1 e^{x^2} dx$. As $1 < e^{x^2} < e^x$ for $x \in (0, 1) \Rightarrow 11 < f(1) < e + 9$.

31^*. Let $f(x) = x - \sin x - a \quad \Rightarrow \quad f'(x) = 1 - \cos x \quad \Rightarrow \quad f'(x) \geq 0 \quad \Rightarrow \quad f(x)$ is increasing
$\Rightarrow f(x) \in \left[f\left(\frac{-\pi}{2}\right), f\left(\frac{\pi}{2}\right)\right] \quad \Rightarrow f(x) \in \left[1 - \frac{\pi}{2}, \frac{\pi}{2} - 1\right]$.

32^*. g(x) is not necessarily be defined in the complete interval [–1, 1] and g(x) will satisfy Rolle's theorem unless $f(x) = 0$ for some $x \in (-1, 1)$.
So far as $f'(x)$ is concerned, without further information nothing definite can be said.

33^*. $f'(x) > 0 \ \forall x > 0$ & $f'(x) < 0 \ \forall x < 0 \quad \Rightarrow \quad f(x) \geq 0 \ \forall x \in R \quad \Rightarrow$ option (A) is true.
$f''(x) = \frac{1}{\sqrt{x^2 + 1}} > 0 \Rightarrow f'(x)$ is increasing.
$f'(x)$ is odd $\quad \Rightarrow f'(-x) = -f'(x) \quad \Rightarrow -f(-x) = -f(x) + c$
$f(0) = 0 \quad \Rightarrow c = 0 \quad \Rightarrow f(-x) = f(x) \quad \Rightarrow f(x)$ is even.

34^*.

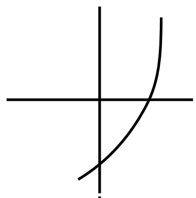


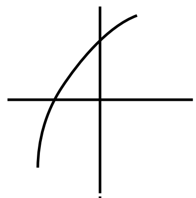
$e^y \frac{dy}{dx} = 2x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2x}{1 + x^2} \quad \Rightarrow \quad m = \frac{2x}{1 + x^2}$

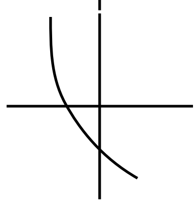
35*. $\frac{\cos^{-1} x - \cos^{-1} y}{x - y}$ = slope of chord = slope of tangent at c (LMVT) = $\frac{-1}{\sqrt{1 - c^2}} \in (-\infty, -1]$

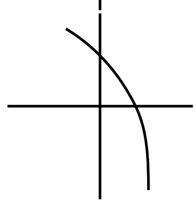
$\Rightarrow \frac{\cos^{-1} x - \cos^{-1} y}{y - x} \geq 1$.

36. $f'(x) = 3 \cos x + 4 \sin x + a > 0$ if $a > 5$
$f'(x) < 0$ if $a < -5$

(A)


(B)


(C)


(D)


39. Let f(x) has degree 'n' then $f'(x)$ has degree (n – 1) and $f''(x)$ has degree (n – 2)

Now, since $f(2x) = f'(x)\, f''(x)$ ...(i)

$\therefore$ $n = (n - 1) + (n - 2)$

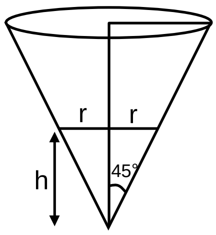
$\therefore$ $\Rightarrow$ $n = 3$

$\therefore$ $\Rightarrow$ $f(x) = ax^3 + bx^2 + cx + d$

Put in equation (i) to get $4a = 9a^2$, $4b = 18ab$, $2c = 6ac + 4b^2$, $d = 2bc$

$\Rightarrow a = 4/9,\ b = c = d = 0 \Rightarrow f(x) = \dfrac{4x^3}{9}$.

40.



$\dfrac{dv}{dt} = -2 \quad \Rightarrow \quad \dfrac{d}{dt}\left(\dfrac{1}{3}\pi r^2 h\right) = -2 \Rightarrow \dfrac{d}{dt}\left(\dfrac{1}{3}\pi r^3\right) = -2 \quad \Rightarrow \quad \pi r^2 \dfrac{dr}{dt} = -2$ ...(i)

Also $\dfrac{d}{dt}(2\pi r) = 2\pi \dfrac{dr}{dt} = 2\pi \left(\dfrac{-2}{\pi r^2}\right) = \dfrac{-4}{r^2}$. At r = 2, $\dfrac{d}{dt}(2\pi r) = \dfrac{-4}{4} = -1 \Rightarrow d = -1$